A. Motivation - $ax = b$ has $x = a^{-1}b$
   a) What does it mean to say the real number a has an inverse?
   b) Do all real numbers have inverses?
   c) Will all matrices have inverses?

B. Definition: An nxn matrix A is said to be invertible if there is a matrix B so that $AB = BA = I$. B is called the inverse of A and is denoted by $A^{-1}$.

C. Example: $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

   a) Proposition: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$ and in this case $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

   b) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ does not have an inverse - $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x+z & y+w \\ 2x+2z & 2y+2w \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

D. Theorem: The inverse of an nxn matrix is unique.
   Proof: Suppose $AB = BA = I$ and $AC = CA = I$. Then $CAB = CI$ or $B = C$.

E. Theorem: An nxn matrix A is invertible iff the system of equations $A\mathbf{x}=\mathbf{b}$ has exactly one solution for every $\mathbf{b}$.

   Proof: Easy part first. If A is invertible then the equation $A\mathbf{x}=\mathbf{b}$ has one solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any b. So now suppose the system of equations has exactly one solution for every $\mathbf{b}$. Let $\mathbf{c}_i$ be the solution to $A\mathbf{x} = \mathbf{e}_i$ and then let $C = [\mathbf{c}_1\ \mathbf{c}_2\ \ldots\ \mathbf{c}_n]$. We then have $AC = I$. Now I must show $CA = I$. To begin this let $\mathbf{x}_i = C\mathbf{a}_I$ for i=1,2, . . . ,n. Then $A\mathbf{x}_i = AC\mathbf{a}_I = I\mathbf{a}_I = \mathbf{a}_I$. Since $A\mathbf{e}_I = \mathbf{a}_I$, we must have $\mathbf{x}_I=\mathbf{e}_I$.
   So now $CA = C[\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n] = [C\mathbf{a}_1\ C\mathbf{a}_2\ \ldots\ C\mathbf{a}_n] = [\mathbf{e}_1\ \mathbf{e}_2\ \ldots\ \mathbf{e}_n] = I$. This means A is invertible and C is its inverse.

F. Procedure for finding $A^{-1}$
   a) Augment $[A \mid I]$ b) Compute $\text{rref}([A \mid I]) = [I \mid A^{-1}]$

G. Definition: An elementary matrix is one obtained by performing exactly one row operation to I.

H. Examples:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

I. If E is an elementary matrix and A is any mxn matrix, then EA is the matrix obtained by performing the elementary row operation to A that was used to form E.

   Proof: Look at $E\mathbf{x}$ for each of the three types for E. $EA = [E\mathbf{a}_1\ E\mathbf{a}_2\ \ldots\ E\mathbf{a}_n]$

J. Theorem: Every elementary matrix is invertible and its inverse is the elementary matrix obtained by performing the "inverse" row operation to I.

K. Theorem: A and B are row equivalent iff there are elementary matrices $E_1, E_2, \ldots, E_k$ so that $A = E_k E_{k-1} \ldots E_3 E_2 E_1 B$.

L. The following statements are equivalent for a square matrix A:

1)* A is invertible
2)* A is row equivalent to $I$ (rref(A)= $I$)
3)* A is a product of elementary matrices
4) $AB = I$ for some B
5) $CA = I$ for some C
6) The columns of A are linearly independent
7) The rows of A are linearly independent
8) The columns of A span $\mathbb{R}^n$.
9) The vectors from the rows of A span $\mathbb{R}^n$.
10)* The system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$
11)* $A\mathbf{x} = \mathbf{0}$ has only the trial solution $\mathbf{x} = \mathbf{0}$

M. Properties of Matrix Inverses

1) $(AB)^{-1} = B^{-1}A^{-1}$ if both A and B are invertible square matrices
2) If A is invertible then $(A^{-1})^{-1} = A$
3) $(cA)^{-1} = \frac{1}{c}A^{-1}$ if A is invertible and $c \neq 0$
4) If A is invertible and n is a positive integer then $\left(A^n\right)^{-1} = \left(A^{-1}\right)^n$ which we write as $A^{-n}$.
5) If A is an invertible matrix and $c \neq 0$ then for any integers m and n we have
   a) $A^m A^n = A^{m+n}$ b) $(A^m)^n = A^{mn}$
   c) $(cA)^n = c^n A^n$
6) If A is invertible then so is $A^T$ and $(A^T)^{-1} = (A^{-1})^T$

N. Theorem: If A is an invertible matrix and AB = AC then B = C and CA = BA implies C=B.

**Assignment: Page 177-179/1,4,6,13,17,19,22,24,26,34,36,38,40,43,44,52**